The Solution of the Initial Mixed Boundary Value Problem for Hyperbolic Equations by Monte Carlo and Probability Difference Methods

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Abstract. The initial mixed boundary value problem for equations of hyperbolic type is considered. It is solved by algorithms "random walk on spheres", "random walk on balls" and "random walk on lattices" of Monte Carlo methods and by probability difference methods.

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1. Introduction

We consider in bounded closed domain $\Omega \in \mathbb{R}^n$, $(n = 2, 3)$ with boundary $\partial\Omega$ and for $t \in (0, T)$ the initial mixed boundary value problem

$$\partial_t^2 u(t,x) - \Delta_x u(t,x) + \gamma^2 u(t,x) = f(t,x), \quad (t,x) \in \Omega \times (0,T), \tag{1.1}$$

$$u(0,x) = \varphi(x), \quad x \in \Omega, \tag{1.2}$$

$$\partial_t u(0,x) = \psi(x), \quad x \in \Omega, \tag{1.3}$$

$$\alpha(t,x)u(t,x) + \beta(t,x)\frac{\partial u(t,x)}{\partial \mathbf{n}} = g(t,x), \quad (t,x) \in \partial\Omega \times (0,T), \tag{1.4}$$

where γ is parameter, $f(t,x)$, $\varphi(x)$, $\psi(x)$, $\alpha(t,x)$, $\beta(t,x)$, $g(t,x)$ are defined functions, $\mathbf{n}$ is normal to $\partial\Omega$. [1], [2], [3]. In monograph [1] the initial value problem for equation (1.1) considered. In AIP Conference Preceedings [2] the mixed problem for elliptic equation by Monte Carlo and by probability difference methods is solved. In article [3] the initial Neumann boundary value problem for parabolic type equation by algorithm "random walk" of Monte Carlo methods is solved.

The initial mixed boundary value problem (1.1) – (1.4) after discretization only on time variable t for $n = 3$ with approximation error $O(\tau^2)$ for equation (1.1), we have:

$$\frac{u^{i+1}(x) - 2u^i(x) + u^{i-1}(x)}{\tau^2} - \Delta u^{i+1}(x) + \gamma^2 u^{i+1}(x) = f^i(x), \tag{1.5}$$

$$i = 1, 2, \ldots, M-1, \quad \tau = \frac{T}{M}, \quad x \in \Omega,$$

$$u^0(x) = \varphi(x), \quad x \in \Omega, \tag{1.6}$$

$$\frac{u^1(x) - u^0(x)}{\tau} = \psi(x), \quad x \in \Omega, \tag{1.7}$$

$$\alpha^{i+1}(x)u^{i+1}(x) + \beta^{i+1}(x)\frac{\partial u^{i+1}(x)}{\partial \mathbf{n}} = g^{i+1}(x), \quad i = 1, 2, \ldots, M-1, \quad x \in \Omega. \tag{1.8}$$

Or

$$\mathbf{L}u^{i+1}(x) = F^i\big(f^i(x), u^{i-1}(x), u^i(x)\big), \quad i = 1, 2, \ldots, M-1, \quad x \in \Omega, \tag{1.9}$$

$$\alpha^{i+1}(x)u^{i+1}(x) + \beta^{i+1}(x)\frac{\partial u^{i+1}(x)}{\partial \mathbf{n}} = g^{i+1}(x), \quad i = 1, 2, \ldots, M-1, \quad x \in \Omega, \tag{1.10}$$

where $\mathbf{L} \equiv \big(\Delta - \tau^2\gamma^2\big)$ is elliptic (Helmholtz) operator.

2. Monte Carlo methods

The main idea of the Monte Carlo methods: we construct the probability value or the probability process in such way that the mean value is the solution of the given problem. Then, as rule, the variance is the precision of the solution. From problems (1.9), 1.10) to exist integral equation second type

$$u(x) = \int_\Omega k(y, x)u(y)dy + v(x), \quad x, y \in \Omega, \tag{2.1}$$

where $v(x)$ is given function, $k(y, x)$ is kernel also given function. The integral equation (2.1) can be is solved Monte Carlo methods, if the integral operator $K$of this equation satisfies the condition

$$\|K\|_{L^1(\Omega)} < 1. \tag{2.2}$$

It the condition (1.10) holds then the integral equation (2.1) can be solved by "random walk on spheres" and "random walk on balls " algorithm of Monte Carlo methods, also it is possible to construct the ε–displaced estimations for $u(x)$, [4], [2], [5], [6], [9], [10].

Is solved by algorithms "random walk on spheres" and "random walk on lattices", [2], [5], [6], of Monte Carlo methods and probability difference methods. [11].

Let $\partial\Omega$ be a Lyapunov surface, the surface Ω be convex. Then the norm of the integral operator acting in $C(\overline{\Omega})$ less than 1. Hence, it is possible to apply Neumann–Ulam scheme to the equation (1.9). [9].

The integral equation (2.1) is solved "random walk on spheres" and by "random walk on balls" algorithm of Monte Carlo methods. [10]. By reaching ε–boundary Markov chain reflected with the probability $p_{\partial\Omega_\varepsilon} = \frac{|\beta^{i+1}(x)|}{|\alpha^{i+1}(x)|+|\beta^{i+1}(x)|}$ and adsorbed with the probability $q_{\partial\Omega_\varepsilon} = \frac{|\alpha^{i+1}(x)|}{|\alpha^{i+1}(x)|+|\beta^{i+1}(x)|}$.

At transition from one condition to the following condition the "weight" of node, that is defined by the recurrence relation

$$Q_0 = 1, \quad Q_{i+1} = Q_i \, \frac{k(x_i, x_{i+1})}{p_\Omega(x_i, x_{i+1})}, \quad i = 0, 1, \ldots,$$

is taken into account. On a border the "weight" of border proportional

$$Q_{\partial\Omega} = \frac{g^{i+1}(x)}{|\alpha^{i+1}(x)| + |\beta^{i+1}(x)|}$$

is taken into account.

Let us denote by h a step of the difference scheme in each coordinate direction and by e_i coordinate unit vector in i-th coordinate direction. We approximate the domain Ω and operator $\mathbf{L}$ by finite difference method. $p^h(x, x \pm e_i h)$, $p^h(x, x + e_i h \pm e_j h)$, $p^h(x, x - e_i h \pm e_j h)$ and $p^h(x, y) = 0$ for the others $x, y \in \Omega \in R_h^n$. Function $p^h(x, y)$ is nonnegative, the sum on y is equal 1 for each x. This means that $p^h(x, y)$ is the probability of transitions of some Markov chain that we denote by $\{\xi_n^h\}$. $p^h(x, y)$ will be coefficients in finite difference approximation.

3. Probability difference method

We'll divide a discrete border into the reflecting $\partial\Omega_R^h$ and the absorbing $\partial\Omega_A^h$, then it is possible to construct the ε–displaced approximation of the unique decision in the point x. For example, it is possible to construct Markov chain by "random walk on lattices" and to define $\{\xi_n^h\}$ along this chain.

Let the set $\partial\Omega_R^h$ approximate $\partial\Omega$ "from within". That is either $x \in \overline{\Omega} \bigcap R_h^3$ or $x \in \partial\Omega$ or straight line connecting x with one of the nearest node $x_i \pm e_i h$, $x_i \pm e_i h \pm e_j h$ or $x_i \pm e_i h \mp e_j h$ touches $\partial\Omega$. The set is determined in $\overline{\Omega} \bigcap R_h^3$. Let's define digitization $\Omega_h = \Omega \bigcap R_h^3 - \partial\Omega_R^h$ of interior Ω and digitization of a stopping set $\partial\Omega_A^h = R_h^3 - \Omega_h - \partial\Omega_R^h$. Then p^h gives transitive probabilities of the approximating chain $\{\xi_i^h\}$ in Ω_h. The chain breaks at the first contact with $\partial\Omega_A^h$. Let's notice that $\mathbf{E}_x\left\{\xi_{n+1}^h - \xi_n^h | \, \xi_n^h = y_i \in \partial\Omega_R^h\right\} = \upsilon(y)\, h/|\upsilon(y)|$. It is coordinated that the reflection from the point $\partial\Omega_R^h$ happens along direction $\upsilon(y)$. $\upsilon(y)$ is the direction of hit into interior node. [2], [5], [6], [11].

After approximation of domains Ω, $\partial\Omega$ and (1.9), (1.10) we receive the problem by finite difference method

$$\mathbf{L}^h u_h^{i+1} = F_h^i\big(f_h^i, u_h^{i-1}, u_h^i\big), \tag{3.1}$$

$$\alpha_h^{i+1} u_h^{i+1} + \beta_h^{i+1} \delta_h\big(u_h^{i+1}\big) = g_h^{i+1}, \tag{3.2}$$

where $\mathbf{L}^h$ is finite difference approximation of the operator $\mathbf{L}$, δ_h is finite difference approximation of the operator $\frac{\partial}{\partial \mathbf{n}}$.

Theorem 1. *It is easy to prove that to finite difference mixed problem (3.1) – (3.2) the Neumann–Ulam scheme is applicable.*

Proof. Let's consider difference problem (1.5) – (1.7) and approximate it by x. We prove the Theorem for case $\Omega \in \mathbb{R}^1 \equiv [0,1]$. Then with step $h = 1/N$ by x we divide $[0,1]$ on N parts. Then we derive the following finite-difference problem for $k = 1, ..., N-1$

$$h^2 u_k^{i+1} - 2h^2 u_k^{i+1} + h^2 u_k^{i-1} - \tau^2 u_{k+1}^{i+1} + 2\tau^2 u_k^{i+1} - \tau^2 u_{k-1}^{i+1} + h^2\tau^2\gamma^2 u_k^{i+1} = h^2\tau^2 f_k^i. \tag{3.3}$$

First order of approximation $\partial_t u(x,0)$ is $O(\tau^2)$. For that we use the following obvious equalities: $\partial_\tau u(x,0) = du(x,0)/dt = \psi(x)$ $t = 0$, $u(x,0) = u_0(x) = \varphi(x)$ and

$$\partial_\tau u(x,0) = \frac{du(x,0)}{dt} = \frac{u(x,\tau) - u(x,0)}{\tau} = \partial_t u(x,0) + \frac{\tau}{2}\partial_t^2 u(x,0) + O(\tau^2). \tag{3.4}$$

From (1.1) when $t = 0$ and using first initial condition we get:

$$\partial_t^2 u(x,0) = \partial_x^2 u(x,0) - \gamma^2 u(x,0) + f(x,0) = \frac{d^2\varphi(x)}{dx^2} - \gamma^2\varphi(x) + f(x,0). \tag{3.5}$$

Now from (3.5) we derive

$$\frac{\tau}{2}\partial_t^2 u(x,0) = \frac{\tau}{2}\Big(\frac{d^2\varphi(x)}{dx^2} - \gamma^2\varphi(x) + f(x,0)\Big). \tag{3.6}$$

Expression $\tau\partial_t^2 u(x,0)/2$ from (3.6) we put in (3.4), and get

$$\partial_\tau u(x,0) - \frac{\tau}{2}\Big(\frac{d^2\varphi(x)}{dx^2} - \gamma^2\varphi(x) + f(x,0)\Big) = \partial_\tau u(x,0) + O(\tau^2).$$

Hence

$$\partial_\tau u(x,0) \approx \frac{u_k^1 - u_k^0}{\tau} = \psi_k + \frac{\tau}{2}\Big(\frac{\varphi_{k+1} - 2\varphi_k + \varphi_{k-1}}{h^2} - \gamma^2\varphi_k + f_k\Big) + O(\tau^2)$$

or

$$u_k^1 = \varphi_k + \tau\psi_k + \frac{\tau}{2}\Big(\frac{\varphi_{k+1} - 2\varphi_k + \varphi_{k-1}}{h^2} - \gamma^2\varphi_k + f_k\Big) + O(\tau^2) \tag{3.7}$$

Thus, we show, that $\partial_t u(x,0)$ (second initial condition) also is approximated with accuracy of $O(\tau^2)$. Finite-difference equation (3.7) with boundary condition (1.8) on every time layer $(i+1)$ is solved by sweep method. Indeed, from (3.3) for $i = 1, 2, \ldots, M-1$, $k = 1, 2, \ldots, N-1$ we get

$$-\tau^2 u_{k+1}^{i+1} + \big(2\tau^2 + h^2 + h^2\tau^2\gamma^2\big)u_k^{i+1} - \tau^2 u_{k-1}^{i+1} = 2h^2 u_k^i - h^2 u_k^{i-1} + h^2\tau^2 f_k^i. \tag{3.8}$$

Sufficient condition of convergent for sweep method of system (3.8) because the inequality is true: $|2\tau^2 + h^2 + h^2\tau^2\gamma^2| > |-\tau^2| + |-\tau^2|$, where τ is step by time t, $\tau > 0$, h is step by space variable x, $h > 0$, and parameter $\gamma^2 > 0$. Thus, solution of

system (3.8) exists and it canbe solved numerically by sweep method. Now, (3.8) is written for each $(i+1)$ in matrix form

$$\mathbf{A}\mathbf{u} = \mathbf{F}, \tag{3.9}$$

where matrix $\mathbf{A}$ is with 3 diagonals: diagonal elements are $2\tau^2 + h^2 + h^2\tau^2\gamma^2$, upper and lower diagonal elements are $-\tau^2$, the rest elements are zeros. Column vector $\mathbf{F}$ is known because of right side of equation, initial and boundary conditions $\mathbf{F} = \mathbf{F}(u_k^i, u_k^{i-1}, f_k^i)$, column vector $\mathbf{u}$ is unknown vector to be found. Solution of system (3.9) one can be written in form:

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{F}. \tag{3.10}$$

Reffering to work [7] we can write out all eigenvalues of matrix $\mathbf{A}$:

$$\lambda_k\big(\mathbf{A}\big) = 2\tau^2 + h^2 + h^2\tau^2\gamma^2 - 2\sqrt{\tau^2\tau^2}\cos(k\vartheta), \;\; k = 1, 2, \ldots, N, \;\; \vartheta = \frac{\pi}{N+1}.$$

If $N \gg 1$, then $\lambda_k\big(\mathbf{A}\big) \geq h^2 + h^2\tau^2\gamma^2 > 0$. From condition $|\lambda_k\big(\mathbf{A}\big)| < 1$ we get relationship between steps τ and h:

$$\frac{-\sqrt{1-h^2}}{h\gamma} < \tau < \frac{\sqrt{1-h^2}}{h\gamma}.$$

When it is considered that $\tau > 0$ and for $\gamma > 0$ we get the condition (Courant type condition)

$$\tau < \frac{\sqrt{1-h^2}}{h\gamma}. \tag{3.11}$$

Analogous researches were done in work [8]. If condition (3.11) is true, then iterational process for solution of system (3.9) (ex., Jacobi method) converges. Existing of solution of system (3.9) (the same for system (3.8)), convergence of numerical method of sweep (inversion of matrix $\mathbf{A}$) for that system, as well as the discrete solution are constructed on absorbing Markov's chain, that terminates on ε-border, since for (3.9) iterational process converges and (1.8) is true (that mixed boundary condition (1.8) consists of two parts: Dirichlet's and Neumann's; due to Dirichlet's condition Markov's chainis absorbed on ε-border, that is terminates. This proves the usability of Neumann–Ulam's scheme for solution of (3.8). This system can be solved by "random walk on lattices", that is probability difference mathod. Theorem is proved.

Now we prove usability of Neumann–Ulam's scheme to (3.8) differently. For that we write (3.8) on time layers $(i+1)$ in form

$$\mathbf{u_r} = \mathbf{K}\mathbf{u_l} + \mathbf{\Phi}, \tag{3.12}$$

where $\mathbf{K}$ is two-diagonal matrix (operator): upper and lowe diagonal elements are $\tau^2/(2\tau^2 + h^2 + h^2\tau^2\gamma^2)$; the rest elements are zeros, $\mathbf{u_r} = \mathbf{u_r}(\mathbf{u}_k^{i+1})$, $\mathbf{u_l} = \mathbf{u_l}(\mathbf{u}_{k+1}^{i+1}, \mathbf{u}_{k-1}^{i+1})$, $\mathbf{\Phi} = \mathbf{\Phi}\big(\mathbf{u}_k^i, \mathbf{u}_k^{i-1}, \mathbf{f}_k^i\big)$. Now show that operator (matrix) $\mathbf{K}$ is compressive operator. For that we calkulate the eigenvalues of matrix $\lambda_k(\mathbf{K})$ of matrix

K, [7]:

$$\lambda_k\big(\mathbf{K}\big) = \frac{-2\tau^2}{2\tau^2 + h^2 + h^2\tau^2\gamma^2}\cos(k\vartheta), \quad k = 1, 2, \ldots, N, \quad \vartheta = \frac{\pi}{N+1}.$$

Easy to see, that for $\tau > 0,\ h > 0,\ \gamma^2 > 0$ condition for spectral radius of matrix **K**

$$\rho\big(\mathbf{K}\big) < 1 \tag{3.13}$$

is true for any $\tau > 0$ and $h > 0$. In that case iterational process (by k, where $k = 0$ and for $k = N$ boundary conditions are true) converges for (3.12) also, condition (3.13) now is necessary and sufficient condition of convergence of iterational process for (3.12). It means, that we can use scheme of Neumann–Ulam for system (3.12), that is that system (and (3.8) also) might be solved by probability difference methods: by constructing converging to ε-border the Markov's chains and all trajectories terminats on ε-border since on borders Dirichlet's conditions presence. Theorem is proved completely. □

Analogous algorithms can be found in works [8], [9], [10].

Algorithm of constructing of Markov's chains from work [11] allows us to construct Markov's chains, and estimate discrete solution of system (3.8):

$$u^h = \mathbf{E}_x\left[\sum_{i=0}^{N_h-1} C_i^h f\big(\xi_i^h\big)\triangle t_i^h I_{\Omega^h}\big(\xi_i^h\big) + C_{N_h-1}^h r\big(\xi_{N_h}^h\big) + \sum_{i=0}^{N_h-1} C_i^h\Big(-g\big(\xi_i^h\big)\Big)d\mu_i^h\right]$$

is the unique discrete solution of the problem (the same discrete solution of problem (3.1), (3.2)), where $\mathbf{E}_x$ is an expectation.

Remark 2. Common error of probability difference method depends on the following parametres: 1) τ step by time t, $\tau > 0$, as $O(\tau^2)$ 2) h step by space variable x, $h > 0$, as $O(h^2)$ 3) parameter $\gamma^2 > 0$ and 4) it depends on ε-border linearly.

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